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One-Sided Approximation in L of the Characteristic Function of an Interval by Trigonometric Polynomials

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Abstract—The value of the best one-sided integral approximation of the characteristic function of the interval $(-h, h)$ by trigonometric polynomials of given degree is found for any $0 < h \le \pi$.
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1. INTRODUCTION AND FORMULATION OF THE MAIN RESULT

In what follows, we use the following notation:

$\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ is the period of length 2π, i.e., a half-open interval $[\alpha, \alpha + 2\pi)$ with identified endpoints, where α is an arbitrary fixed number[3] from $\mathbb{R}$;

$L = L(\mathbb{T})$ is the space of 2π-periodic measurable real-valued functions with the norm $\|f\| = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{T}} |f(t)|\,dt$;

$\mathcal{T}_n$ is the subspace of trigonometric polynomials $\tau(x) = a_0 + \sum\limits_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)$ of degree at most n with real coefficients; and

$$E_n(g) := \inf_{\tau\in\mathcal{T}_n} \|g - \tau\|, \qquad E_n^-(g) := \inf_{\tau\in\mathcal{T}_n,\, \tau\le g} \|g - \tau\|, \qquad E_n^+(g) := \inf_{\tau\in\mathcal{T}_n,\, g\le\tau} \|g - \tau\|$$

are values of the best integral approximation and the best integral approximation from below and from above, respectively, of a bounded function $g \in L$ by the subspace $\mathcal{T}_n$.

For an arbitrary fixed number $h \in (0, \pi]$, denote by χ_h the characteristic function of the interval $(-h, h)$ periodically extended to $\mathbb{R}$ with period 2π. We set

$$\mathcal{E}_n(h) := E_n(\chi_h), \qquad \mathcal{E}_n^-(h) := E_n^-(\chi_h), \qquad \mathcal{E}_n^+(h) := E_n^+(\chi_h).$$

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[3]In the present paper, we take $-\pi$ or 0 for α.

Note that

$$\mathcal{E}_n^-(h) = \mathcal{E}_n^+(\pi - h) \quad \text{for} \quad n \in \mathbb{N}, \quad h \in (0, \pi). \tag{1.1}$$

Hence, using the classical Fejér's result (1913) about the exact constant in the inequality between the uniform and integral norms of a nonnegative trigonometric polynomial of degree at most n (see [9, Ch. 2, Sect. 6, Subsect. 7, Problem 50]), we obtain

$$\lim_{h \to \pi} \mathcal{E}_n^-(h) = \lim_{h \to 0} \mathcal{E}_n^+(h) = \frac{1}{n+1}.$$

In what follows, in view of (1.1), we investigate only the value $\mathcal{E}_n^-(h)$.

The problem of one-sided approximation of the function $\operatorname{sgn} x$ and of the characteristic function of an open interval by entire functions of exponential type and by trigonometric polynomials was studied in connection with applications in number theory by Beurling, Selberg, and Vaaler. It is established [19] (see also [13, Ch. 1]) that

$$\mathcal{E}_n^-(h) \le \frac{1}{n+1} \quad \text{for any} \quad n \in \mathbb{N}, \quad h \in (0, \pi]. \tag{1.2}$$

In the present paper, we continue the investigation initiated in [3], where an integral approximation of the characteristic function χ_h by trigonometric polynomials of given degree is found for any $h \in (0, \pi]$. Here, we obtain a similar final result in the case of one-sided integral approximation.

Let $\lfloor \beta \rfloor = \max\{\nu \in \mathbb{Z} : \ \nu \le \beta\}$ and $\lceil \beta \rceil = \min\{\nu \in \mathbb{Z} : \ \beta \le \nu\}$. Introduce the functions

$$\Lambda_n(x) = \frac{\sin \dfrac{nx}{2} - \sin \dfrac{(n+2)x}{2}}{(n+2) \sin \dfrac{nx}{2} - n \sin \dfrac{(n+2)x}{2}}, \tag{1.3}$$

$$\lambda_n(x) = \frac{\sin x}{\left\lceil \dfrac{n}{2} \right\rceil \sin x - \sin\left(\left\lceil \dfrac{n}{2} \right\rceil x\right) \cos\left(\left\lfloor \dfrac{n+2}{2} \right\rfloor x\right)}, \tag{1.4}$$

$$\gamma_n(x) = \frac{\sin x}{\left\lceil \dfrac{n}{2} \right\rceil \sin x + \sin\left(\left\lfloor \dfrac{n+2}{2} \right\rfloor x\right) \cos\left(\left\lceil \dfrac{n}{2} \right\rceil x\right)}. \tag{1.5}$$

Let us formulate the result of this paper, in which we use the notation

$$I_j = I_{j,n} = \left(\frac{j\pi}{n+1}, \frac{(j+1)\pi}{n+1}\right), \quad n \in \mathbb{N}, \quad j = 0, 1, \dots, n. \tag{1.6}$$

Theorem 1. *Let $n \in \mathbb{N}$. Then, the following statements are valid:*

(*a*) $\mathcal{E}_n^-(h) = \dfrac{1}{n+1}$ *for* $h = \dfrac{j\pi}{n+1}$, $j = 1, \dots, n+1$;

(*b*) $\mathcal{E}_n^-(h) = \dfrac{h}{\pi}$ *for* $h \in I_{0,n}$;

(*c*) $\mathcal{E}_n^-(h) = \dfrac{h}{\pi} - \Lambda_n(h)$ *for* $h \in I_{1,n}$;

(*d*) $\mathcal{E}_n^-(h) = \dfrac{h}{\pi} - \displaystyle\sum_{k=0}^{j-1} \gamma_n(x_k)$ *for* $h \in I_{2j,n}$, $j = 1, \dots, \lfloor n/2 \rfloor$, $n \ge 2$,

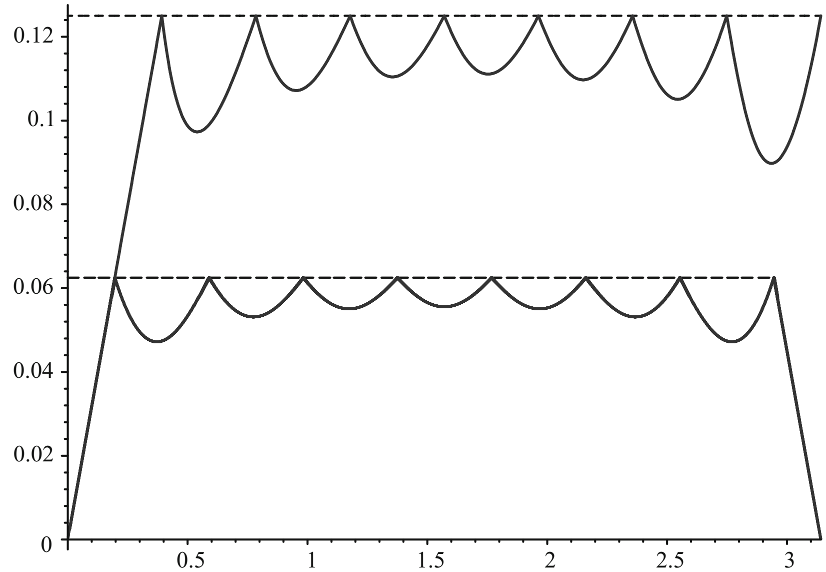

Fig. 1. Graphs of the functions $\mathcal{E}_7^-(h)$ and $\mathcal{E}_7(h)$ of the variable $h \in (0, \pi]$.

where $x_0, \ldots, x_{j-1}$ *are zeros of the function* $\cos\dfrac{nh}{2}\cos\dfrac{(n+2)x}{2} - \cos\dfrac{(n+2)h}{2}\cos\dfrac{nx}{2}$ *located in the interval* $(0, h)$ *in ascending order*;

$$(e)\quad \mathcal{E}_n^-(h) = \frac{h}{\pi} - \Lambda_n(h) - \sum_{k=0}^{j-1}\lambda_n(y_k) \quad \textit{for} \quad h \in I_{2j+1,n}, \quad j = 1, \ldots, \lceil n/2 \rceil - 1, \quad n \geq 3,$$

where $y_0, \ldots, y_{j-1}$ *are zeros of the function* $\sin\dfrac{nh}{2}\sin\dfrac{(n+2)x}{2} - \sin\dfrac{(n+2)h}{2}\sin\dfrac{nx}{2}$ *located in the interval* $(0, h)$ *in ascending order.*

Figure 1 shows graphs of the functions $\mathcal{E}_n^-(h)$ and $\mathcal{E}_n(h)$ of the variable $h \in (0, \pi]$ as well as graphs of the constant functions $\dfrac{1}{n+1}$ and $\dfrac{1}{2(n+1)}$ (the dashed lines) for the fixed value of the parameter $n = 7$; the graph of the function $\mathcal{E}_7(h)$ is taken from [3].

2. SOME TYPES OF CONDITIONS FOR THE HERMITE INTERPOLATION OF THE FUNCTION χ_h: THE FIRST MAIN LEMMA

We present several conditions of interpolation of the function χ_h by cosine polynomials on $[0, \pi]$. Let

$$s, r \in \{0, 1\}, \quad \ell, m \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}, \quad s + \ell \geq 1.$$

To an ordered quadruple of numbers (s, ℓ, m, r), we assign the type $T(s, \ell, m, r)$ of conditions of interpolation of the function χ_h by a cosine polynomial τ of minimal possible degree. This type characterizes the location of interpolation nodes and their multiplicity. Let us describe the menaning of each of the parameters s, ℓ, m, r:

(a) if $s = 0$, then 0 is not an interpolation node;

(b) if $s = 1$, then 0 is an interpolation node; i.e., $\tau(0) = 1$;

(c) ℓ is the number[4] of interpolation nodes $x_1 < x_2 < \ldots < x_\ell$ located in the open interval $(0, h)$;

[4]If $\ell = 0$, then there are no interpolation nodes in $(0, h)$. Similarly, in condition (e), if $m = 0$, then there are no interpolation nodes in (h, π).

each of these nodes has multiplicity 2:

$$\tau(x_j) = 1, \quad \tau'(x_j) = 0, \quad j = 1, \dots, \ell; \tag{2.1}$$

(d) the point $x_{\ell+1} := h$ always is a (simple) interpolation node; i.e., $\tau(h) = 0$;

(e) m is the number of interpolation nodes $x_{\ell+2} < x_{\ell+3} < \dots < x_{\ell+m+1}$ located in the interval (h, π); each of these nodes has multiplicity 2:

$$\tau(x_j) = 0, \quad \tau'(x_j) = 0, \quad j = \ell+2, \dots, \ell+m+1; \tag{2.2}$$

(f) if $r = 0$, then π is not an interpolation node;

(g) if $r = 1$, then π is an interpolation node; i.e., $\tau(\pi) = 0$.

For brevity, we will call a cosine polynomial τ (of minimal possible degree) providing a type $T(s, \ell, m, r)$ of conditions of Hermite interpolation of the function χ_h a *cosine polynomial of type* $T(s, \ell, m, r)$; sometimes, we will say that the *cosine polynomial* τ *has type* $T(s, \ell, m, r)$. The degree of this polynomial is equal to the number of interpolation conditions minus one; i.e.,

$$\deg \tau = s + r + 2(\ell + m). \tag{2.3}$$

Using the change of variable $t = \cos x$, we pass to the corresponding problem of Hermite interpolation of the characteristic function of the half-open interval $(\cos h, 1]$ by an algebraic polynomial $P(t)$ connected with the cosine polynomial $\tau(x)$ by the relation

$$P(\cos x) = \tau(x), \quad x \in [0, \pi]. \tag{2.4}$$

It is known that, in the case of more general interpolation conditions,[5] the algebraic Hermite polynomial (of minimal possible degree) exists and is unique. Explicit formulas for this polynomial and its degree are given in [4, Ch. 1, Sect. 11]. In the case when all interpolation nodes have multiplicity 2, these formulas have a rather simple form (see [4, Ch. 1, Sect. 11, Subsect. 1, formulas (17), (24)]).

In order to estimate the value $\mathcal{E}_n^-(h)$ from above and construct the corresponding extremal cosine polynomial, we need a lemma, which, for a polynomial τ of type $T(0, \ell, m, 0)$ (more precisely, for the algebraic polynomial P connected with the polynomial τ by relation (2.4)), is a special case of a result established independently in the 1880s by Markov [8, Paper 1] and Stieltjes [17, 18] (see [12, Sect. 3.411], Lemmas 9 and 9′ in [10, Ch. 1, Sect. 1.2], and Lemmas 9 and 9′ in [11, Sect. “Tauberian Theory and Its Applications,” Subsect. 12, pp. 312–314]). The general case, for cosine polynomials τ of type $T(s, \ell, m, r)$, is proved similarly. To make the presentation complete, we give the following statement with proof.

Lemma 1. *Assume that* $s, r \in \{0, 1\}$, $\ell, m \in \mathbb{Z}_+$, $s + \ell \geq 1$, $h \in (0, \pi]$, *and* τ *is an interpolation cosine polynomial* (*of minimal possible degree*) *providing the type* $T(s, \ell, m, r)$ *of interpolation conditions for the function* χ_h. *Then,*

$$\tau(x) \leq \chi_h(x) \quad \textit{for all} \quad x \in [-\pi, \pi]. \tag{2.5}$$

Proof. Denote by n the degree of the cosine polynomial τ. Using formula (2.3), we find $n = s + r + 2(\ell + m)$. The derivative of the polynomial τ is representable in the form of the product $\tau'(x) = (\sin x)\theta(x)$, where θ is some cosine polynomial of degree $\deg \theta = n - 1 = s + r + 2(\ell + m) - 1$.

[5] At each node, the function and its several successive derivatives are interpolated. and the number of interpolated values of the derivatives depends on the index of the node.

The derivative $\tau'(x)$ vanishes at the endpoints of the interval $[0,\pi]$. Let us count the number of pairwise distinct zeros of the derivative $\tau'(x)$ located inside the open interval $(0,\pi)$. By (2.1) and (2.2), we have

$$\tau'(x_j)=0 \quad \text{for} \quad j=1,\dots,\ell \quad \text{and} \quad j=\ell+2,\dots,\ell+m+1;$$

moreover, the nodes x_j lie inside the interval $(0,\pi)$ and their total number is $\ell+m$.

The number of intervals (x_j,x_{j+1}) at the endpoints of which the polynomial τ takes the same values (either unit or zero) is $s+\ell+m+r-1$; here, we take into account that the point $x_{\ell+1}:=h$ is always an interpolation node; i.e., $\tau(h)=0$. The derivative $\tau'(x)$ vanishes inside any such interval. Thus, the total number of pairwise distinct zeros of the derivative $\tau'(x)$ inside the open interval $(0,\pi)$ is at least $d:=s+r+2(\ell+m)-1$. The number d coincides with the degree of the cosine polynomial θ; i.e., $d=\deg\theta$. Therefore, the derivative τ' has no zeros in $[0,\pi]$ distinct from the zeros listed above. Denote the zeros of the derivative τ' in $[0,\pi]$ as follows: $y_0:=0<y_1<\ldots<y_d<y_{d+1}:=\pi$.

By the assumption of the lemma, $s+\ell\geq 1$. This implies that there is at least one interpolation node in $[0,h)$; denote by x^* the rightmost of these nodes ($x^*=0$ if $\ell=0$ and $x^*=x_\ell$ if $\ell\geq 1$).

Since $\tau(x^*)=1$ and $\tau(h)=0$, the derivative $\tau'(x)$ is negative in the interval (x^*,h). From this, we can uniquely identify the sign of the derivative on any interval

$$(y_j,y_{j+1}), \quad j=0,\dots,d,$$

as well as points of local minima and maxima of the polynomial τ. In particular, x^* is a point of local maximum. Analyzing the remaining points of local extrema of the polynomial τ lying in the interval $[0,\pi]$, we come to inequality (2.5). The lemma is proved. $\square$

As an example, consider a cosine polynomial τ of type $T(0,\ell,m,0)$.

Denote by $\vartheta(x)$ a Hermite interpolation cosine polynomial that interpolates with multiplicity 2 the function

$$(\cos x-\cos h)\chi_h(x)$$

at nodes $x_1<\ldots<x_\ell<x_{\ell+1}=h<x_{\ell+2}<\ldots<x_{\ell+m+1}$ located in the open interval $(0,\pi)$ (we set $\vartheta(h)=\vartheta'(h)=0$ at the node $x_{\ell+1}=h$). An explicit form of the polynomial $\vartheta(x)$ can be easily found with the help of formulas (17) and (24) from [4, Ch. 1, Sect. 11, Subsect. 1]. It is easy to check that the cosine polynomial $\tau(x)=\dfrac{\vartheta(x)}{\cos x-\cos h}$ has type $T(0,\ell,m,0)$.

Below (see Fig. 2 at the end of Section 6), a cosine polynomial $\tau_h\in\mathcal{T}_{10}$ of type $T(0,2,3,0)$ is constructed by this method, which is a polynomial of the best integral approximation from below of the function χ_h for $h=9\pi/22$.

3. GAUSS TYPE QUADRATURE FORMULA FOR TRIGONOMETRIC POLYNOMIALS: PROOF OF STATEMENT (b) OF THEOREM 1

The following quadrature formula is well known (see [5, Vol. 2, Ch. 10, formula (2.5)]):

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\tau(x)\,dx=\frac{1}{n+1}\sum_{k=0}^{n}\tau\left(\xi+\frac{2k\pi}{n+1}\right); \tag{3.1}$$

it holds for an arbitrary polynomial $\tau\in\mathcal{T}_n$ and any fixed $\xi\in\mathbb{R}$.

For $\xi = \pi/(n+1)$, formula (3.1) takes the form

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \tau(x)\,dx = \frac{1}{n+1}\sum_{k=0}^{n} \tau\left(t_k\right), \quad t_k = \frac{(2k+1)\pi}{n+1},$$

where τ is an arbitrary polynomial from $\mathcal{T}_n$. From this formula, the following statement is easily obtained.

Corollary. *Let* $\tau \in \mathcal{T}_n$ *and* $\tau(x) \le 0$ *for* $\dfrac{\pi}{n+1} \le |x| \le \pi$. *Then,* $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \tau(x)\,dx \le 0$.

Proof of statement (*b*) of Theorem 1. Assume that $0 < h \le \dfrac{\pi}{n+1}$ and τ is an arbitrary polynomial from $\mathcal{T}_n$ satisfying the inequality $\tau(x) \le \chi_h(x)$ for all $x \in [-\pi, \pi]$. Then, $\tau(x) \le 0$ for $h \le |x| \le \pi$. Since $0 < h \le \dfrac{\pi}{n+1}$, we have $\tau(x) \le 0$ for $\dfrac{\pi}{n+1} \le |x| \le \pi$. By Corollary, $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \tau(x)\,dx \le 0$. Hence, we obtain

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \{\chi_h(x) - \tau(x)\}\,dx = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \chi_h(x)\,dx - \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \tau(x)\,dx \ge \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \chi_h(x)\,dx = \frac{h}{\pi}.$$

Therefore, the lower estimate $\mathcal{E}_n^-(h) \ge h/\pi$ is valid. The polynomial $\tau(x) \equiv 0$ implies the upper estimate $\mathcal{E}_n^-(h) \le h/\pi$. Statement (*b*) of Theorem 1 is proved.

4. THE SECOND MAIN LEMMA: PROOF OF STATEMENT (*a*) OF THEOREM 1

Let us present a statement (Lemma 2), which will be used below to derive a lower estimate of the required value $\mathcal{E}_n^-(h)$ for $\pi/(n+1) < h < \pi$. The proof of Lemma 2 in the case of a continuous function g is contained in [6, Ch. 1, Sect. 1.7, Theorem 1.7.5] and is carried over to the case of an arbitrary bounded function $g \in L$ almost word-for-word.

Lemma 2. *Assume that the quadrature formula*

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} u(x)\,dx \approx \sum_{k=1}^{m} p_k u(x_k)$$

with nonnegative coefficients $p_1, \ldots, p_m$ *is exact on* $\mathcal{T}_n$. *Then, for any bounded function* $g \in L$,

$$E_n^-(g) \ge \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} g(x)\,dx - \sum_{k=1}^{m} p_k g(x_k).$$

For an arbitrary fixed number $h \in (0, \pi]$, denote by $\chi_{(0,2h)}$ the characteristic function of the interval $(0, 2h)$ periodically extended to $\mathbb{R}$ with period 2π. Since the subspace $\mathcal{T}_n$ is invariant with respect to any translation, we have

$$\mathcal{E}_n^-(h) := E_n^-(\chi_h) = E_n^-(\chi_{(0,2h)}). \tag{4.1}$$

We apply Lemma 2 to obtain a lower estimate for the value

$$\mathcal{E}_n^-(h_j), \qquad h_j = \frac{j\pi}{n+1}, \qquad j = 1, \ldots, n+1,$$

which, together with upper estimate (1.2), will allow us to prove statement (a) of Theorem 1.

Note that statement (a) of Theorem 1 is a simple corollary of the result by Markov and Stieltjes cited above (see the paragraph before Lemma 1 in Section 2), an analog of Lemma 2 for $E_n^+(g)$, and relation (1.1). However, to make the presentation complete, we give the proof.

Proof of statement (a) of Theorem 1. Let $h_j = \dfrac{j\pi}{n+1}$, $j = 1, \dots, n+1$. For $\xi = 0$, formula (3.1) takes the form

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \tau(x)\,dx = \frac{1}{n+1}\sum_{k=0}^{n} \tau(x_k), \quad x_k = \frac{2k\pi}{n+1}, \quad \tau \in \mathcal{T}_n.$$

Hence, using (4.1) and Lemma 2, we obtain the lower estimate

$$\mathcal{E}_n^-(h_j) = E_n^-(\chi_{(0,2h_j)}) \ge \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \chi_{(0,2h_j)}(x)\,dx - \frac{1}{n+1}\sum_{k=0}^{n} \chi_{(0,2h_j)}(x_k) = \frac{1}{n+1},$$

which, together with (1.2), implies statement (a) of Theorem 1.

5. QUADRATURE FORMULAS WITH SEVERAL FIXED NODES

The proof of statements (c)–(e) of Theorem 1 is based on Lemmas 1 and 2. In turn, to apply these lemmas, we need quadrature formulas with several fixed nodes and of the highest degree of precision. Such formulas for algebraic polynomials have been studied for a long time, starting with investigations by Gauss, Christoffel, Lobatto, Mehler, and Markov. The history of this subject is given in monograph [12]. Explicit formulas for coefficients of these quadrature formulas (in particular, signs of the coefficients) and the arrangement of nodes are of great interest. This is closely related to polynomial interpolation–orthogonal bases and orthogonal polynomials. Fejér, Steklov, Erdös, Turán, Winston, Schohat, and others obtained important results in this field (a partial description of the results and the corresponding bibliography can be found in [2, 14, 15]).

Steklov [16] (see [15, Sect. 2, Theorem 3]) proved that a quadrature formula of the highest degree of precision with one fixed node (independently of its location) has only positive coefficients; however, one of the free nodes in this formula can lie outside the interval of integration. Winston [20, Sect. 3] found explicit formulas for coefficients of the corresponding quadrature formulas of the highest degree of precision with fixed nodes in the case when they lie on the boundary or outside the interval of integration for an integral with generic weight. Schohat [15, Sect. 2, Subsect. 4] found similar formulas for the case when some fixed nodes lie inside the interval of integration.

We need quadrature formulas expressing the mean value of a cosine polynomial on the period in terms of a linear combination of its values at points from the interval $[0, \pi]$. More precisely, we need quadrature formulas with a given number of nodes (some of them are fixed); moreover, these quadrature formulas must have the highest degree of precision; i.e., they must be valid for cosine polynomials of maximal possible degree. For our purposes, four types of specified quadrature formulas are sufficient: a formula with one fixed node h; two formulas with two fixed nodes 0, h and h, π; and one formula with three fixed nodes 0, h, and π. To construct these formulas, we use a special case of a known statement given in [15, Sect. 2, Theorem 1] (see also [7, Ch. 9]) in terms of algebraic polynomials; however, using the cosine change, we can easily reformulate this statement in terms of cosine polynomials.

Theorem 2. *Assume that points $\alpha_1 < \alpha_2 < \ldots < \alpha_m$ belong to the interval $[0,\pi]$ and points $x_1 < x_2 < \ldots < x_\nu$ belong to the set $[0,\pi]\backslash\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$. Then, numbers $A_1,\ldots,A_m,B_1,\ldots,B_\nu$ such that*

$$\frac{1}{\pi}\int\limits_0^\pi \tau(x)\,dx = \sum_{\ell=1}^m A_\ell\tau(\alpha_\ell) + \sum_{k=1}^\nu B_k\tau(x_k) \tag{5.1}$$

for any cosine polynomial τ of degree at most $2\nu+m-1$ exist if and only if the polynomial

$$(\cos x - \cos x_1)(\cos x - \cos x_2)\cdots(\cos x - \cos x_\nu) \tag{5.2}$$

is orthogonal to any cosine polynomial of degree at most $\nu-1$ with weight

$$(\cos x - \cos\alpha_1)(\cos x - \cos\alpha_2)\cdots(\cos x - \cos\alpha_m).$$

Note that, if formula (5.1) is valid for any cosine polynomial $\tau \in \mathcal{T}_{2\nu+m-1}$, then

$$\frac{1}{2\pi}\int\limits_{-\pi}^\pi f(x)\,dx = \sum_{\ell=1}^m A_\ell\frac{f(\alpha_\ell)+f(-\alpha_\ell)}{2} + \sum_{k=1}^\nu B_k\frac{f(x_k)+f(-x_k)}{2} \tag{5.3}$$

for an arbitrary trigonometric polynomial $f \in \mathcal{T}_{2\nu+m-1}$.

Recall some known facts and terms related to quadrature formula (5.1). The points $\alpha_1,\ldots,\alpha_m$ and $x_1,\ldots,x_\nu$ are called *fixed* and *free* nodes of quadrature formula (5.1), respectively; the numbers A_ℓ, B_k are its *coefficients*.

Thus, one can vary the parameters $A_1,\ldots,A_m$, $B_1,\ldots,B_\nu$, $x_1,\ldots,x_\nu$ to maximize the dimension of the space of cosine polynomials for which formula (5.1) is exact. In contrast to these parameters, the numbers $\alpha_1,\ldots,\alpha_m$ are fixed and cannot be varied.

Since the number of free parameters is $2\nu+m$, it is natural to suppose that we can choose them so that quadrature formula (5.1) be exact for an arbitrary cosine polynomial of degree $2\nu+m-1$ (because the number of coefficients of such a polynomial is $2\nu+m$). This maximal degree of a polynomial is called *the highest degree of precision* of quadrature formula (5.1).

Note that Theorem 2 can be reformulated in the following equivalent form: *the cosine polynomial*

$$\psi(x) = (\cos x - \cos x_1)\cdots(\cos x - \cos x_\nu)(\cos x - \cos\alpha_1)\cdots(\cos x - \cos\alpha_m) \tag{5.4}$$

of degree $\nu+m$ must be orthogonal to any cosine polynomial of degree $\le \nu-1$ with unit weight.

This means that the coefficients with indices $0,\ldots,\nu-1$ in the decomposition of the polynomial ψ in cosines must be zero; i.e., this polynomial is a linear combination of successive harmonics with indices $\nu,\ldots,\nu+m$. In other words, the polynomial ψ, along with representation (5.4), must admit the following representation:

$$\psi(x) = a_\nu\cos\nu x + a_{\nu+1}\cos(\nu+1)x + \ldots + a_{\nu+m}\cos(\nu+m)x. \tag{5.5}$$

Note that, by (5.4), the polynomial ψ vanishes at all nodes of the quadrature formula including the fixed points $\alpha_1,\ldots,\alpha_m$. Hence, using (5.5), we obtain one more important representation for ψ in the determinant form

$$\psi(x) = \begin{vmatrix} c_\nu(\alpha_1) & c_{\nu+1}(\alpha_1) & \ldots & c_{\nu+m}(\alpha_1) \\ c_\nu(\alpha_2) & c_{\nu+1}(\alpha_2) & \ldots & c_{\nu+m}(\alpha_2) \\ \ldots & \ldots & \ldots & \ldots \\ c_\nu(\alpha_m) & c_{\nu+1}(\alpha_m) & \ldots & c_{\nu+m}(\alpha_m) \\ c_\nu(x) & c_{\nu+1}(x) & \ldots & c_{\nu+m}(x) \end{vmatrix};$$

here,

$$c_j(x) := \cos jx.$$

Note the connection of the quadrature formulas under investigation with Christoffel–Darboux kernels (see [12]). For this, consider a special case of quadrature formula (5.1) with one fixed node α and ν free nodes $x_1, \ldots, x_\nu$. In this case, the highest degree of precision is $2\nu + 1 - 1 = 2\nu$ and the polynomial ψ takes the form

$$\psi(x) = \begin{vmatrix} c_\nu(\alpha) & c_{\nu+1}(\alpha) \\ c_\nu(x) & c_{\nu+1}(x) \end{vmatrix} = \cos\nu\alpha\cos(\nu+1)x - \cos(\nu+1)\alpha\cos\nu x.$$

It is clear that the required free nodes $x_1, \ldots, x_\nu$ coincide with zeros of the fraction

$$\frac{\psi(x)}{\cos x - \cos\alpha} = \frac{\cos\nu\alpha\cos(\nu+1)x - \cos(\nu+1)\alpha\cos\nu x}{\cos x - \cos\alpha}.$$

This fraction is the Christoffel–Darboux kernel

$$K_\nu(\alpha, x) = 1 + 2\sum_{j=1}^{\nu}\cos j\alpha\cos jx = \frac{\cos\nu\alpha\cos(\nu+1)x - \cos(\nu+1)\alpha\cos\nu x}{\cos x - \cos y}$$

for the system $\{1, \sqrt{2}\cos x, \sqrt{2}\cos 2x, \ldots\}$, which is orthonormal with respect to the scalar product $(f, g) := \dfrac{1}{\pi}\displaystyle\int_0^\pi f(x)g(x)\,dx$.

For a quadrature formula of the highest degree of precision with several fixed nodes, we have a similar connection with the Christoffel–Darboux kernel for a system of cosine polynomials orthonormal with respect to the weighted scalar product $(f, g)_v := \displaystyle\int_0^\pi f(x)g(x)v(x)\,dx$, where the function v is expressed in terms of the fixed nodes.[6]

6. THE PROOF OF STATEMENTS (c)–(e) OF THEOREM 1

The proof of Theorem 1 is based on quadrature formulas of the highest degree of precision. For the integral approximation from below of the function χ_h by polynomials of even degree $n = 2\nu$, we use quadrature formulas with one fixed node h and three fixed nodes $0, h$, and π. These formulas allow us to obtain Theorem 1 in the case of approximation by polynomials of even degree for h belonging to even intervals I_{2k} and odd intervals I_{2k+1}, respectively (see (1.6)). The case of even intervals is considered in Lemma 3; the case of odd intervals is considered in Lemma 4. Similarly, for the approximation of χ_h from below by polynomials of odd degree, we use Lemmas 5 and 6 with two fixed nodes $0, h$ and h, π, respectively. In Lemmas 3–6, we use the notion of cosine polynomial of type $T(s, \ell, m, r)$ introduced in Section 2.

Lemma 3. *Let $n = 2\nu$, $\nu \in \mathbb{N}$, and $x_0 \in I_{0,n}$. Then, the following statements hold*:

(a) *the polynomial*

$$\cos\frac{nx_0}{2}\cos\frac{(n+2)x}{2} - \cos\frac{(n+2)x_0}{2}\cos\frac{nx}{2} = \cos\nu x_0\cos(\nu+1)x - \cos(\nu+1)x_0\cos\nu x$$

has exactly $1 + n/2$ *zeros* $x_0 < x_1 < \ldots < x_{n/2}$ *in* $(0, \pi)$;

[6] If some fixed nodes lie inside the interval $(0, \pi)$, then the function v is alternating.

(*b*) *the zero* $x_k = x_k(x_0)$ *with index* $k = 0, 1, \dots, n/2$ *increases and runs over the interval* $I_{2k,n}$ *when* x_0 *runs over the interval* $I_{0,n}$;

(*c*) *for any* $\tau \in \mathcal{T}_n$, *the following quadrature formula is valid*:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \tau(x)\,dx = \sum_{k=0}^{n/2} \gamma_n(x_k)\frac{\tau(x_k)+\tau(-x_k)}{2}, \tag{6.1}$$

where the function $\gamma_n(x)$ *is defined by* (1.5); *moreover,* $\gamma_n(x) > 0$ *for any* $x \in (0,\pi)$;

(*d*) *for arbitrary* $k = 1, \dots, n/2$ *and any* $h \in I_{2k,n}$, *the polynomial*

$$\cos\frac{nh}{2}\cos\frac{(n+2)x}{2} - \cos\frac{(n+2)h}{2}\cos\frac{nx}{2}$$

has exactly $1 + n/2$ *zeros* $x_0 < x_1 < \ldots < x_{n/2}$ *in the open interval* $(0,\pi)$; *moreover,* $x_k = h$ *and there exists a cosine polynomial* $\tau_h \in \mathcal{T}_n$ *of type* $T(0,k,m,0)$, $k+m = n/2$, *satisfying the following conditions*: $\tau_h(x) \le \chi_h(x)$ *for all* $x \in \mathbb{R}$ *and* $\tau_h(x_s) = \chi_h(x_s)$ *for* $s = 0, 1, \dots, n/2$.

Proof. As mentioned above, there is a known method for constructing quadrature formulas of the highest degree of precision with given number of fixed nodes and given number of free nodes (see [20, Sect. 3; 15, Sect. 2, Subsect. 4]). The method is based on Christoffel–Darboux kernels (see [12]).

To make the presentation complete, we establish quadrature formula (6.1) of the highest degree of precision with one fixed node x_0 and prove the statements of Lemma 3 that characterize the properties of coefficients and nodes of this quadrature formula.

In the case under consideration, the Christoffel–Darboux kernel

$$K_\nu(y,x) = 1 + 2\sum_{j=1}^{\nu}\cos jy\cos jx = \frac{\cos\nu y\cos(\nu+1)x - \cos(\nu+1)y\cos\nu x}{\cos x - \cos y} \tag{6.2}$$

is constructed for the cosine system $\{1, \sqrt{2}\cos x, \sqrt{2}\cos 2x, \dots\}$ orthonormal with respect to the scalar product

$$(f,g) := \frac{1}{\pi}\int_0^{\pi} f(x)g(x)\,dx.$$

Using (6.2) and l'Hôpital's rule, we obtain the known relations

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_\nu(y,x)\,dx = 1, \tag{6.3}$$

$$0 < K_\nu(y,y) = 1 + 2\sum_{j=1}^{\nu}\cos^2 jy = \frac{(\nu+1)\cos\nu y\sin(\nu+1)y - \nu\cos(\nu+1)y\sin\nu y}{\sin y}; \tag{6.4}$$

for $y = 0$ and π, we assume the value of the fraction in (6.4) to be equal to $2\nu+1$.

Required cosine polynomial (5.2) is Christoffel–Darboux kernel (6.2) with fixed value of the first variable $y = x_0$:

$$K_\nu(x_0,x) = \frac{\cos\nu x_0\cos(\nu+1)x - \cos(\nu+1)x_0\cos\nu x}{\cos x - \cos x_0} = 1 + 2\sum_{j=1}^{\nu}\cos jx_0\cos jx. \tag{6.5}$$

Quadrature formula (5.3) corresponding to the case under consideration takes the following form on $\mathcal{T}_{2\nu}$:

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\tau(x)\,dx=\gamma_0\frac{\tau(x_0)+\tau(-x_0)}{2}+\sum_{k=1}^{\nu}\gamma_k\frac{\tau(x_k)+\tau(-x_k)}{2},\tag{6.6}$$

where $x_1,\dots,x_\nu$ are pairwise distinct zeros of polynomial (6.5) located in $[0,\pi]$.

In [2, Sect. 4], it is shown that the condition $\left|\dfrac{\cos(\nu+1)x_0}{\cos\nu x_0}\right|\le 1$ is necessary and sufficient for the existence of these zeros. In addition, for any $x_0\in\left[0,\dfrac{\pi}{2\nu+1}\right]$, polynomial (6.5) has exactly ν zeros $x_1<x_2<\ldots<x_\nu$ in $[0,\pi]$; moreover, the zero $x_k=x_k(x_0)$ with index $k=1,\dots,\nu$ monotonically increases and runs over the interval $\left[\dfrac{2k\pi}{2\nu+1},\dfrac{(2k+1)\pi}{2\nu+1}\right]$ when x_0 changes from 0 to $\dfrac{\pi}{2\nu+1}$.

Let us calculate the coefficients of quadrature formula (6.6). We set

$$w(x):=\cos\nu x_0\cos(\nu+1)x-\cos(\nu+1)x_0\cos\nu x,$$

$$w_k(x):=\frac{w(x)}{\cos x-\cos x_k}=\frac{\cos\nu x_0\cos(\nu+1)x-\cos(\nu+1)x_0\cos\nu x}{\cos x-\cos x_k},\qquad k=0,\dots,\nu.$$

Let $x_0\in[0,\pi]$ and $\cos\nu x_0\ne 0$. Then, according to Theorem 1 from [2, Sect. 2], the equality

$$\frac{K_\nu(x_k,x)}{K_\nu(x_k,x_k)}=\frac{w_k(x)}{w_k(x_k)}$$

holds for all $k=0,\dots,\nu$; consequently,

$$\gamma_k=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{K_\nu(x_k,x)}{K_\nu(x_k,x_k)}\,dx.$$

Hence, using (6.3) and (6.4), we obtain

$$\begin{aligned}0<\gamma_k=\frac{1}{K_\nu(x_k,x_k)}&=\frac{\sin x_k}{(\nu+1)\cos\nu x_k\sin(\nu+1)x_k-\nu\cos(\nu+1)x_k\sin\nu x_k}\\&=\frac{\sin x_k}{\left\lceil\frac{n}{2}\right\rceil\sin x_k+\sin\left(\left\lfloor\frac{n+2}{2}\right\rfloor x_k\right)\cos\left(\left\lceil\frac{n}{2}\right\rceil x_k\right)},\qquad k=0,\dots,\nu.\end{aligned}$$

Thus, statements (a)–(c) are proved.

Statement (d) follows from statement (b) and Lemma 1. □

Lemma 4. *Let $n=2\nu$, $\nu\in\mathbb{N}$, and $x_0\in I_{1,n}$. Then, the following statements hold:*

(a) *the polynomial*

$$\sin\frac{nx_0}{2}\sin\frac{(n+2)x}{2}-\sin\frac{(n+2)x_0}{2}\sin\frac{nx}{2}$$

has exactly $1+n/2$ zeros in $(0,\pi]$: $x_0<x_1<\ldots<x_{n/2}=\pi$;

(b) *the zero $x_k=x_k(x_0)$ with index $k=0,\dots,n/2-1$ increases and runs over the interval $I_{2k+1,n}$ when x_0 runs over $I_{1,n}$;*

(*c*) *for any* $\tau \in \mathcal{T}_n$, *the following quadrature formula is valid*:

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \tau(x)\,dx = \Lambda_n(x_0)\tau(0) + \sum_{k=0}^{n/2} \lambda_n(x_k)\,\frac{\tau(x_k)+\tau(-x_k)}{2}, \tag{6.7}$$

where the function $\Lambda_n(x)$ *is defined by formula* (1.3) *and the function* $\lambda_n(x)$ *is defined by formula* (1.4) *for* $x \in (0,\pi)$ *and* $\lambda_n(\pi) = \Lambda_n(\pi - x_0)$; *moreover, all the coefficients* $\Lambda_n(x_0), \lambda_n(x_0), \lambda_n(x_1), \ldots, \lambda_n(x_{n/2})$ *are nonnegative*;

(*d*) *for arbitrary* $k = 0, \ldots, n/2-1$ *and any* $h \in I_{2k+1,n}$, *the polynomial*

$$\sin\frac{nh}{2}\sin\frac{(n+2)x}{2} - \sin\frac{(n+2)h}{2}\sin\frac{nx}{2}$$

has exactly $1 + n/2$ *zeros* $x_0 < x_1 < \ldots < x_{n/2} = \pi$ *in* $(0,\pi]$; *moreover,* $x_k = h$ *and there exists a cosine polynomial* $\tau_h \in \mathcal{T}_n$ *of type* $T(1,k,m,1)$, $k+m = n/2-1$, *satisfying the following conditions*: $\tau_h(x) \le \chi_h(x)$ *for all* $x \in \mathbb{R}$, $\tau_h(0) = \chi_h(0)$, *and* $\tau_h(x_s) = \chi_h(x_s)$ *for* $s = 0, \ldots, n/2$.

Let us present a quadrature formula with fixed nodes 0 and x_0 (Lemma 5) and a quadrature formula with fixed nodes x_0 and π (Lemma 6), which are exact on the set of trigonometric polynomials of odd degree $n = 2\nu - 1$. Recall that $\lfloor \xi \rfloor$ means the maximal integer not exceeding ξ.

Lemma 5. *Let* $n = 2\nu - 1$, $\nu \in \mathbb{N}$, *and* $x_0 \in I_{1,n}$. *Then, the following statements hold*:

(*a*) *the function*

$$\sin\frac{nx_0}{2}\sin\frac{n+2}{2}x - \sin\frac{n+2}{2}x_0\sin\frac{nx}{2}$$

has exactly $1 + \lfloor n/2 \rfloor$ *zeros in the interval* $(0,\pi)$: $x_0 < x_1 < \ldots < x_{\lfloor n/2 \rfloor}$;

(*b*) *the zero* $x_k = x_k(x_0)$ *with index* $k = 0, \ldots, \lfloor n/2 \rfloor$ *increases and runs over the interval* $I_{2k+1,n}$ *when* x_0 *runs over* $I_{1,n}$;

(*c*) *for any* $\tau \in \mathcal{T}_n$, *the following quadrature formula is valid*:

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \tau(x)\,dx = \Lambda_n(x_0)\tau(0) + \sum_{k=0}^{\lfloor n/2 \rfloor} \lambda_n(x_k)\,\frac{\tau(x_k)+\tau(-x_k)}{2}, \tag{6.8}$$

where the functions $\Lambda_n(x)$ *and* $\lambda_n(x)$ *are defined by formulas* (1.3) *and* (1.4), *respectively*; *moreover, all the coefficients* $\Lambda_n(x_0)$, $\lambda_n(x_0), \ldots, \lambda_n(x_{\lfloor n/2 \rfloor})$ *are nonnegative*;

(*d*) *for arbitrary* $k = 0, \ldots, \lfloor n/2 \rfloor$ *and any* $h \in I_{2k+1,n}$, *the function*

$$\sin\frac{nh}{2}\sin\frac{n+2}{2}x - \sin\frac{n+2}{2}h\sin\frac{nx}{2}$$

has exactly $\lfloor n/2 \rfloor$ *zeros* $x_0 < x_1 < \ldots < x_{\lfloor n/2 \rfloor}$ *in the interval* $(0,\pi)$; *moreover,* $x_k = h$ *and there exists a cosine polynomial* $\tau_h \in \mathcal{T}_n$ *of type* $T(1,k,m,0)$, $k+m = \lfloor n/2 \rfloor = \nu - 1$, *satisfying the conditions* $\tau_h(x) \le \chi_h(x)$ *for* $x \in \mathbb{R}$, $\tau_h(0) = \chi_h(0)$, *and* $\tau_h(x_s) = \chi_h(x_s)$ *for* $s = 0, \ldots, \lfloor n/2 \rfloor$.

Lemma 6. *Let* $n = 2\nu - 1$, $\nu \in \mathbb{N}$, $\nu \ge 2$, *and* $x_0 \in I_{0,n}$. *Then, the following statements hold*:

(*a*) *the function*

$$\cos\frac{nx_0}{2}\cos\frac{n+2}{2}x - \cos\frac{n+2}{2}x_0\cos\frac{nx}{2}$$

has exactly $1 + \lfloor n/2 \rfloor$ *zeros in the interval* $(0,\pi)$: $x_0 < x_1 < \ldots < x_{\lfloor n/2 \rfloor}$;

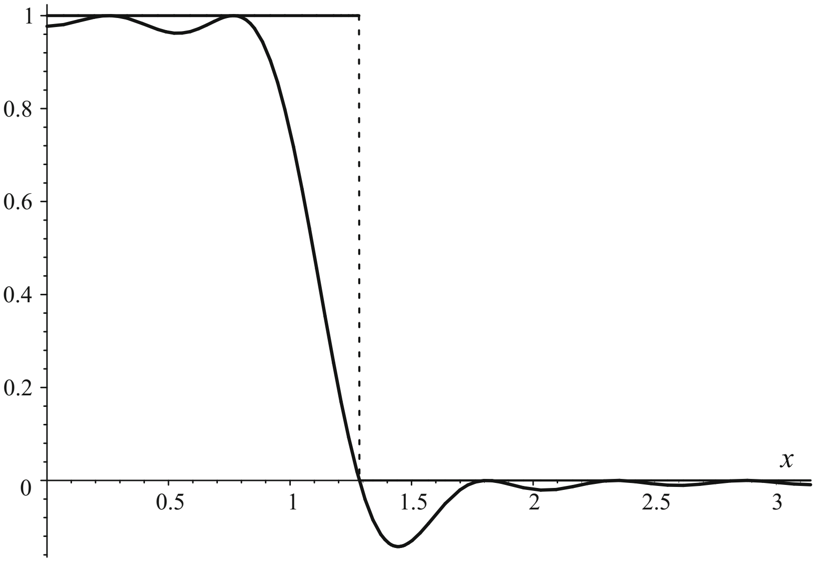

Fig. 2. Graphs of χ_h and $\tau_h \in \mathcal{T}_{10}$ on $[0,\pi]$ for $h = 9\pi/22$.

(*b*) *the zero* $x_k = x_k(x_0)$ *with index* $k = 0, \dots, \lfloor n/2 \rfloor$ *monotonically increases and runs over the interval* $I_{2k,n}$ *when* x_0 *runs over the interval* $I_{0,n}$;

(*c*) *for any* $\tau \in \mathcal{T}_n$, *the following quadrature formula is valid*:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \tau(x)\,dx = \Lambda_n(\pi - x_0)\tau(\pi) + \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_n(x_k)\,\frac{\tau(x_k)+\tau(-x_k)}{2}, \tag{6.9}$$

where the functions $\Lambda_n(x)$ *and* $\gamma_n(x)$ *are defined by formulas* (1.3) *and* (1.5), *respectively*; *moreover, all the coefficients* $\Lambda_n(\pi - x_0)$, $\gamma_n(x_0), \dots, \gamma_n(x_{\lfloor n/2 \rfloor})$ *are nonnegative*;

(*d*) *for arbitrary* $k = 1, \dots, \lfloor n/2 \rfloor$ *and any* $h \in I_{2k,n}$, *the function*

$$\cos\frac{nh}{2}\cos\frac{n+2}{2}x - \cos\frac{n+2}{2}h\cos\frac{nx}{2}$$

has exactly $1 + \lfloor n/2 \rfloor$ *zeros* $x_0 < x_1 < \dots < x_{\lfloor n/2 \rfloor}$ *in the interval* $(0,\pi)$; *moreover,* $x_k = h$ *and there exists a cosine polynomial* $\tau_h \in \mathcal{T}_n$ *of type* $T(0,k,m,1)$, $k+m = \lfloor n/2 \rfloor = \nu - 1$, *satisfying the conditions* $\tau_h(x) \le \chi_h(x)$ *for all* $x \in \mathbb{R}$, $\tau_h(\pi) = \chi_h(\pi)$, *and* $\tau_h(x_s) = \chi_h(x_s)$ *for* $s = 0, \dots, \lfloor n/2 \rfloor$.

Lemmas 4, 5, and 6 are proved similarly to Lemma 3. Note also that statements (*a*) and (*c*) of Lemmas 4 and 5 are contained in a more general statement [1, Lemma 1]; in addition, Lemma 6 can be proved by using Lemma 5 and replacing x by $\pi - x$.

Figure 2 shows graphs of the function χ_h and the cosine polynomial $\tau_h \in \mathcal{T}_{10}$ of its best integral approximation from below on the interval $[0,\pi]$ for $h = 9\pi/22$. The polynomial τ_h is constructed with the use of Lemma 3 and the construction described at the end of Section 2.

The proof of statements (*c*)–(*e*) of Theorem 1 follows from Lemmas 2–6. In more detail, for $h \in I_{2j+1} = I_{2j+1,n}$ (see (1.6)), the lower estimate for the value $\mathcal{E}_n^-(h)$ is obtained on the base of Lemma 2 with application of quadrature formulas (6.7) and (6.8) for even and odd n, respectively. The upper estimate, coinciding with this lower estimate, is given by the cosine polynomials $\tau_h \in \mathcal{T}_n$ presented in statements (*d*) of Lemmas 4 and 5; the deviation of the polynomial τ_h from the function χ_h approximated from below in the integral metric is calculated with the help of quadrature formulas (6.7) and (6.8) for even and odd n, respectively. Similarly, in the case $h \in I_{2j} = I_{2j,n}$, the lower estimate for $\mathcal{E}_n^-(h)$ follows from Lemma 2 and quadrature formulas (6.1) and (6.9); the upper estimate, coinciding with this lower estimate, is obtained with the use of statements (d) of Lemmas 3 and 6.

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